Large N Limit of Higher Derivative Extended CP(N) Model

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Abstract

We construct a fourth-order derivative CP(N) model in $1+1$ dimensions by incorporating the topological charge density squared term into the Lagrangian. We quantize the theory by reformulating with auxiliary fields and then performing the path integral explicitly. We discuss the renormalizability in the large N limit and relevance of the effective action with axion physics.

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1. Introduction.

The $1+1$ dimensional CP(N) model has been studied intensively in the large N limit [1]. It is known to be exactly soluble, and exhibits interesting phenomena such as dynamical mass generation and asymptotic freedom which is also one of the essential properties of gauge theories [1]. Therefore it is important to extend the CP(N) model and investigate various properties. In this paper, we consider a higher derivative extension of the model. More specifically, we invent a fourth-order derivative interaction by utilizing the topological charge density on the CP(N). One of the merit of our higher derivative theory is that it permits an auxiliary field formalism and the path integral can be performed explicitly. The theory has only logarithmic divergence in the large N limit, still it remains non-renormalizable due to the lack of counter terms. But we find that the effective theory is well-defined and describes a two-dimensional axion interacting with Maxwell theory.

We start with the coadjoint orbit approach to nonlinear sigma model [2] for some generality and introduce a coadjoint orbit variable $Q = gKg^{-1}$ on G/H, where $g \in G$ and K which belongs to the Lie algebra $\mathcal{H}$ of H is the centralizer for $\mathcal{H}$. In order to construct higher derivative theory, we consider $t \equiv \epsilon^{\mu\nu} < Q\,\partial_\mu Q\,\partial_\nu Q >$ which is the topological charge density on the coadjoint orbit G/H. The topological charge $T = c_1 \int d^2x\, t$ with some normalization constant c_1 is completely characterized by the homotopy group $\pi_1(H)$. It is well known that this group is in general given by (sum of) additive group of integers depending on the nature of the centralizer K, and the topological charge T characterizes the instanton solution of the nonlinear sigma model described by the field Q [3]. Our higher derivative model is constructed by adding the fourth derivative term t^2 to the nonlinear sigma model :

$$S = \frac{1}{g^2}\int d^2x \left(-\frac{1}{2} < \partial_\mu Q\,\partial^\mu Q > + \frac{t^2}{2M^2} \right), \tag{1}$$

where $< \cdots >$ denotes trace and M is some mass scale in the theory. In SU(2) case, the above model reduces to the two-dimensional low energy effective description of QCD which was proposed by Faddeev and Niemi [4]. Also, it is related with Skyrmion model, because t^2 can be written as $t^2 \sim < J_{\mu\nu} J^{\mu\nu} >$, $J_{\mu\nu} = [\partial_\mu Q, \partial_\nu Q]$.

Let us focus on CP(N) case and study the large N limit of (1) by using the path integral quantization method. We will show that the above higher derivative theory admits an exact path integration in the large N analysis. In the CP(N) orbit, the coadjoint variable Q can be described by a single N component complex column vector $z = (z_1, \cdots, z_N)^T$ which defines the CP(N) $\equiv$ SU(N)/SU($N-1$) $\times$ U(1) manifold [5]:

$$Q = -izz^\dagger + i\frac{I}{N}, \quad z^\dagger z = 1. \tag{2}$$

Then, the topological charge density is given by

$$\epsilon^{\mu\nu} < Q\,\partial_\mu Q\,\partial_\nu Q > = i\epsilon^{\mu\nu}(\partial_\mu z)^\dagger(\partial_\nu z). \tag{3}$$

Note that this is quadratic in z which is essential for the path integral in the large N limit.

Expressing the Lagrangian (1) in terms of $z^\dagger$ and z and introducing two auxiliary fields $A_\mu(x)$ and $b(x)$, we obtain

$$\mathcal{L} = \frac{1}{g^2}\left[(D_\mu z)^\dagger (D^\mu z) + ib\,\epsilon^{\mu\nu}(\partial_\mu z)^\dagger(\partial_\nu z) - \frac{M^2}{2}\,b^2 - \lambda\,(z^\dagger z - 1)\right], \tag{4}$$

where $D_\mu \equiv \partial_\mu - iA_\mu$. Note that z with the constraint $z^\dagger z = 1$ contains $2N-1$ real scalars, whereas the coset space is a $(2N-2)$-dimensional manifold. This mismatch is due to the local U(1) symmetry of the model which is obvious from (1) and (2): Q is invariant under the local U(1) phase rotation of z. Specifically, the Lagrangian is invariant under the following U(1) gauge transformation:

$$z(x) \to e^{i\alpha(x)} z(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x), \quad b(x) \to b(x), \tag{5}$$

of which phase mode $\alpha(x)$ stands for a gauge redundancy in z. Solving the equation of motion and eliminating A_μ and b fields, we see that the extra second and third terms in the Lagrangian actually provide the fourth derivative term $\frac{t^2}{2M^2} = -\frac{1}{2M^2}\left[\epsilon^{\mu\nu}(\partial_\mu z)^\dagger(\partial_\nu z)\right]^2$ with the constraint $z^\dagger z = 1$. Note that even in two dimensions this fourth derivative term is not renormalizable in perturbative sense. The leading divergence is given by a quadratic divergence in one-loop four-points functions. In the large N limit, however, we find that the only divergence is given by a logarithmic divergence.

In order to proceed, we separate the field z into $2N-2$ Nambu-Goldstone bosons $\psi \equiv (z_1, \ldots, z_{N-1})^T$ associated with the spontaneously broken SU(N) symmetry and Higgs bosons $z_N \equiv \frac{g}{\sqrt{2}}(\sigma + i\chi)$. In general there are two possible phases: (I) $\langle\sigma\rangle \neq 0$, $\langle\lambda\rangle = 0$ and (II) $\langle\sigma\rangle = 0$, $\langle\lambda\rangle \neq 0$. In phase (I) both global SU(N) and local U(1) symmetries are broken simultaneously and ψ arise as massless Goldstone bosons. Through the Higgs mechanism χ turns to a longitudinal mode of massive gauge boson A_μ. On the other hand in phase (II) both global SU(N) and local U(1) are not spontaneously broken. Instead ψ and z_N are combined into z with a universal mass $\langle\lambda\rangle^{1/2}$. Note that in two and less than two dimensions any continuous symmetry cannot be broken spontaneously (Coleman-Mermin-Wagner theorem) so that the phase (II) is the only possible phase in two dimensions.

2. Large N effective action

The $1/N$ expansion provides a systematic way of non-perturbative resummation of Feynman diagrams. Even if a given theory is non-renormalizable in perturbative expansion, it possibly can turn into a renormalizable theory through such a resummation technique as the $1/N$ expansion. In fact the CP(N) model is non-renormalizable in larger than two dimensions, whereas the model in less than four dimensions can be renormalized in $1/N$ expansion.

We can rewrite the Lagrangian up to total derivative terms as

$$\mathcal{L} = \frac{1}{g^2}\,z^\dagger\left[-\partial^2 - m^2 - \Gamma\right]z + \frac{\lambda}{g^2} - \frac{M^2}{2g^2}\,b^2, \tag{6}$$

where we separate the Goldstone boson mass m^2 from $\lambda \equiv m^2 + \tilde{\lambda}$ and Γ stands for the interaction terms:

$$\Gamma \equiv -iA_\mu(\partial^\mu - \overleftarrow{\partial^\mu}) - A_\mu A^\mu - ib\,\overleftarrow{\partial^\mu}\epsilon_{\mu\nu}\partial^\nu, \tag{7}$$

where $\overleftarrow{\partial^\mu}$ does not operate on both A_μ and b. The large N effective action is given by

$$S_{\text{eff}} = \int d^2x\, \mathcal{L} + iN\, \mathrm{Tr}\, \mathrm{Ln}\Big[-\partial^2 - m^2\Big] - iN \sum_{n=1}^{\infty} \frac{1}{n}\, \mathrm{Tr}\Big[\frac{1}{-\partial^2 - m^2}\,\Gamma\Big]^n . \tag{8}$$

After some straightforward calculations with momentum cutoff Λ in a gauge invariant way, the effective action up to quadratic terms ($n = 1, 2$) is obtained as

$$\begin{aligned} S_{\text{eff}} = N \int d^2x \Bigg[& \frac{1}{Ng^2}\, z^\dagger \Big[-\partial^2 - m^2 - \Gamma\Big] z - \frac{1}{2}\frac{M^2}{Ng^2}\, b^2 \\ & +(m^2 + \tilde{\lambda}) \left[\frac{1}{Ng^2} - \frac{1}{4\pi}\ln\frac{\Lambda^2}{m^2}\right] - \frac{1}{4\pi}\, m^2 + \frac{1}{2}\,\tilde{\lambda}\, \Pi_\lambda(i\partial)\, \tilde{\lambda} \\ & -\frac{1}{4}\, F_{\mu\nu}\, \Pi_A(i\partial)\, F^{\mu\nu} + \frac{1}{2}\, b\, \Pi_{bA}(i\partial)\, \epsilon^{\mu\nu} F_{\mu\nu} - \frac{1}{4}\, b\, \partial^2 \Pi_{bA}(i\partial)\, b \Bigg]. \end{aligned} \tag{9}$$

where the vacuum polarization functions are given by

$$\Pi_\lambda(p) = \frac{1}{2\pi p^2}\sqrt{\frac{-p^2}{4m^2 - p^2}}\, \ln\left(\frac{\sqrt{4m^2 - p^2} - \sqrt{-p^2}}{\sqrt{4m^2 - p^2} + \sqrt{-p^2}}\right), \tag{10}$$

$$\Pi_A(p) = \frac{1}{\pi p^2} + \left(1 - \frac{4m^2}{p^2}\right)\Pi_\lambda(p), \tag{11}$$

$$\Pi_{bA}(p) = -\frac{1}{4\pi}\ln\frac{\Lambda^2}{m^2} + \frac{1}{4\pi} - \frac{1}{2}\, p^2\, \Pi_A(p). \tag{12}$$

Note that there arise logarithmic divergences in $b\,\partial^2 b$ and $b\,\epsilon^{\mu\nu} F_{\mu\nu}$ terms which have no counter terms in the original Lagrangian.

3. Renormalization in $1/N$ leading order

Renormalization of the coupling g can be worked out in the same manner as in the original CP(N) model. The large N effective potential is defined as the effective action divided by $\Omega \equiv \int d^2x$ with $\tilde{\lambda}$, A_μ, b, z, $z^\dagger$ all set equal to zero. It is given by

$$\frac{1}{N} V_{\text{eff}} = -\frac{m^2}{Ng^2} + \frac{m^2}{4\pi}\left[\ln\frac{\Lambda^2}{m^2} + 1\right], \tag{13}$$

where Ng^2 is fixed finite in the large N limit so that the $1/N$ expansion can be treated systematically as a loop expansion. The Goldstone boson mass m is determined as a nontrivial solution to the gap equation:

$$\frac{dV_{\text{eff}}}{dm^2} = 0 \quad \longleftrightarrow \quad \frac{1}{g^2} = \frac{N}{4\pi}\ln\frac{\Lambda^2}{m^2}, \tag{14}$$

from which m^2 reads

$$m^2 = \Lambda^2 \exp\left[-\frac{4\pi}{Ng^2}\right]. \tag{15}$$

We notice that m can be independent of the ultraviolet (UV) cutoff Λ by imposing Λ dependence on the coupling g. In fact the scale invariance condition $\Lambda\, dm/d\Lambda = 0$ leads us to the Callan-Symanzik β-function

$$\beta(g) \equiv \Lambda \frac{dg}{d\Lambda} = -\frac{Ng^3}{4\pi}, \tag{16}$$

which shows the asymptotically free behavior of the coupling. In the original CP(N) the only divergence is the one which arises in the large N effective action through a tadpole diagram coupled with $\tilde{\lambda}$ so that the scale invariance condition $\Lambda\, dm/d\Lambda = 0$ is enough to achieve the cutoff independent theory. Since m is scale invariant, the gap equation (14) suggests the renormalization of coupling is given by

$$\frac{1}{Ng^2} - \frac{1}{4\pi} \ln \frac{\Lambda^2}{m^2} = \frac{1}{Ng_R^2} - \frac{1}{4\pi} \ln \frac{\mu^2}{m^2}, \tag{17}$$

where g_R is the renormalized coupling at the reference energy scale μ.

In the extended model, however, logarithmic divergences arise in the coefficients of the induced extra terms, $b\,\partial^2 b$ and $b\,\epsilon^{\mu\nu} F_{\mu\nu}$ which do not have their counter terms in the classical action. Therefore our theory is not renormalizable though the coupling g can be renormalized in the same way as in the original CP(N) model. Let us look at how this argument works in the large N effective action (9). The induced kinetic terms of A_μ and $\tilde{\lambda}$ are UV finite in themselves so that we do not need wave function renormalization for them. Then the third term in the right hand side of Eq. (9) becomes UV finite through Eq. (17) from which the Z factor for the coupling can be read

$$Z^{-1} \equiv \frac{g_R^2}{g^2} = 1 + \frac{Ng_R^2}{4\pi} \ln \frac{\Lambda^2}{\mu^2}. \tag{18}$$

The kinetic term of z has to be UV finite in itself so that we see

$$\frac{1}{Ng^2}\, z^\dagger \left[-\partial^2 - m^2\right] z = \frac{1}{Ng_R^2}\, z_R^\dagger \left[-\partial^2 - m^2\right] z_R, \tag{19}$$

where z_R has been introduced through $z = Z_z^{1/2} z_R$ and Z_z is thereby determined as $Z_z \equiv Z$ in order to cancel the Z factor from the coupling renormalization. Thus we realize that Γ in the effective action has to remain invariant through renormalization procedure. This forces all of A_μ, $\tilde{\lambda}$ and b to be unchanged through renormalization. Thus we obtain another renormalization condition

$$\frac{M^2}{Ng^2} = \frac{M_R^2}{Ng_R^2}, \tag{20}$$

which determines the Z factor for M^2 as $Z_M \equiv Z$. On the other hand, we do not have any degrees of freedom to subtract the logarithmic divergences which arise in the kinetic term of b and in the mixing term between b and A_μ. This makes our extended model non-renormalizable even after the $1/N$ resummation.

Even though the theory is not renormalizable, the effective theory is a well defined renormalizable theory. In fact, by using the gap equation (14), the logarithmic Λ dependence of $\Pi_{bA}(i\partial)$ in (9) can be completely eliminated in favor of the $1/N$ counting rule $Ng^2 =$ fixed, and we see that the effective action for A_μ and b field is given by

$$S_{\rm eff}[A_\mu, b] = \frac{1}{g^2}\int d^2x \left[\frac{c}{4}\,(\partial_\mu b)(\partial^\mu b) + \frac{c}{2}\, b\,\epsilon^{\mu\nu}F_{\mu\nu} - \frac{M^2}{2}\, b^2\right] - \frac{N}{48\pi m^2}\int d^2x\, F_{\mu\nu}F^{\mu\nu}, \qquad (21)$$

where we have expanded the vacuum polarization functions with respect to $p^2/4m^2$ after the analytic continuation from $p^2 < 0$ to $0 \le p^2 < 4m^2$. Here, the constant $c \equiv Ng^2/4\pi - 1$ confines the physical region of coupling to $g^2 > 4\pi/N$. This is a two-dimensional model in which the Maxwell field A_μ interacts with a pseudoscalar field b with $b\,{}^*F$ axion type interaction [6]. If a CP violating term $\mathcal{L}_{CP} = \theta\,\epsilon^{\mu\nu} < Q\,\partial_\mu Q\,\partial_\nu Q >$ [7] is present in the original action (1), it can be always absorbed into the $\frac{t^2}{2M^2}$ term by shifting $t \to t + \theta M^2$ up to an irrelevant constant. In the auxiliary field formulation of (4), this shifts the interaction $ib\,\epsilon^{\mu\nu}(\partial_\mu z)^\dagger(\partial_\nu z) \to b\,(i\epsilon^{\mu\nu}(\partial_\mu z)^\dagger(\partial_\nu z) + \theta M^2)$. However, the potential prefers $\langle b\rangle = 0$ for the minimum, and the effect of such CP violating term will be suppressed.

In summary, we have constructed a higher derivative CP(N) model and quantized it by using the path integral method. We have illustrated that in the large N limit, the ultraviolet divergence can be completely isolated into a logarithmic divergence but the theory remains non-renormalizable due to the lack of counter terms. However, we have found that the effective action is a well-defined renormalizable theory, and it describes a massless gauge field interacting with massive axion. It would be interesting to check whether such a composite higher derivative axion model could have some relevance in higher dimensional axion physics.

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